CASI AERO 2021
June 14-18

Practical Formulation of Kane's Method for Finite Element Representation of Rigid-Flexible Systems

Alexander Schock
Ph.D Student
Carleton University
Ottawa, Canada
AlexRSchock@cmail.carleton.ca

Dr. Robert Langlois
Dept. of Mech. and Aero. Engineering
Carleton University
Ottawa, Canada
Robert.Langlois@carleton.ca

***Abstract*—The formulation of the equations of motion for rigid-flexible multibody dynamic systems can become increasingly cumbersome when using finite element representation of flexible bodies. Common methods such as the Lagrange and Newton-Euler methods respectively suffer from the need for extensive partial differentiations and complications associated with constraint force considerations. Kane's method presents an elegant alternative. Yet, this method also suffers from exhaustive consideration of nodal coupling forces along each generalized coordinate with finite element representation of flexible bodies.**

This paper presents an alternative formulation of Kane's method for rigid-flexible systems where the prescription of generalized coordinates relative to centres of mass of rigid bodies is relaxed such that finite element nodal displacements can be used as generalized coordinates in Kane's equation for the representation of flexible bodies. Special attention is paid to frames of reference definition to enable independent development of the linear dynamic finite element matrices for direct incorporation into the Kane's-method-based formulation. The presented formulation isolates the flexible body modelling which enables the choice of finite element modelling technique, and simplifies the modelling of systems comprising flexible and rigid bodies.

The presented formulation is shown to preserve exact rigid-body mass moment of inertia and first moment of mass when the appropriate interpolation shape functions are chosen for the finite element representation. Moreover, the formulation has been successfully implemented and verified in SRAMSS-2D, a planar shipboard helicopter dynamic interface analysis simulation package which models the aircraft using a rigid body airframe and a finite element skid-tube landing gear representation.

***Index Terms*—Kane's method, multibody dynamics, rigid-flexible body, finite element modelling, numerical methods**

I. INTRODUCTION

The formulation of the governing equations of motion (EOM) for a system is the quintessential problem in dynamics. The ever increasing computational power of modern computers has enabled the analysis of growingly complex dynamical systems. Of interest for this work, are the flexible multibody or multibody with flexible elements undergoing large reference motion problems which have garnered significant attention over the past few decades. This class of dynamical problems frequently arises in a wide range of industries; from aircraft and spacecraft, to flexible manipulators, to compliant mechanisms. As a result, it has been the subject of extensive research which has yielded many approaches to obtain the governing EOM. A comprehensive review of these approaches is beyond the scope of this work. Nevertheless Shabana [1] presents an in-depth review of the progress in flexible multibody dynamics up until the turn of the 21st Century.

Among the reviewed discrete representations of flexible bodies, the most widely used approaches include the *floating frame of reference* (FFR) which superimposes the reference motion of a body's coordinate system, and the body's deformation with respect to the body's coordinate system; the *absolute nodal coordinate formulation* (ANCF) which leverages absolute global displacements and slopes to describe nodal coordinates to obtain exact inertial behaviour when using element shape functions; the *intermediate element coordinate system* (IECS), a convected coordinate system which introduces an intermediate frame of reference for each finite element which is aligned to a reference body frame to capture exact inertial phenomena; and the *finite segment method* (FSM) which differs from typical *finite element methods* (FEM) in that a flexible body is discretized into a finite number of rigid bodies that are elastically and viscously coupled, thus enabling the application of rigid body mechanics to deformable bodies.

In any case, the method used to generate the EOM of a system is another independent consideration beyond the selection of flexible body representation. The two most popular methods for the development of EOM are not well suited for the general rigid-flexible multibody problem. For complex systems, the Newton-Euler method may be unsuitable since it is an inefficient maximal-coordinate method where every active and constraint force for every body must be considered. The Lagrangian method is often also unattractive, even if it is a minimal-coordinate method, due to exhaustive partial differentiations for larger more complex systems.

In [2]–[4], Huston uses Kane's method to generate the EOM for the FSM. It is a powerful method for generating the EOM for multibody systems through use of generalized coordinates and partial velocities, and where the resulting equations are easily manipulated into a first-order system which lends itself to numerical computation and state propagation [5], [6]. However, as with any discretization procedure, the size of a system is directly affected by the number of discretized components. Moreover, the exhaustive consideration of coupling forces in the active force term of Kane's method becomes laborious for more complex systems. In [4], Huston suggests the combination of the FSM with FEM techniques to exert the full capabilities of the FSM method.

In this paper, we present a practical approach to the formulation of the EOM of a rigid-flexible system using Kane's method, in which the linear matrices of standard FEM representing the flexible body can be independently developed. The FSM is briefly applied to separate the rigid-flexible system into an open chain of coupled rigid and flexible bodies for treatment using Kane's method. Specifically for the flexible body, Kane's method is relaxed such that Kane's equation can be applied on the nodes of the finite element representation to describe flexible body motion. With careful definition of the frames of reference, a system definition similar to the IECS approach can be obtained where rigid body mass moments of inertia can be preserved [7] while satisfying FEM element assembly constraints. Constraint equations are developed to express the coupling between the rigid and flexible bodies such that the final minimal set of EOM can be obtained for the rigid-flexible body.

This paper is organized into five sections. The development of this formulation begins in Section II with the specific definitions, and mathematical notations and conventions that are required. Then, Kane's method is described in Section III. The definitions and manipulations of Kane's method required for the incorporation of the FSM and IECS are developed in Section IV. Discussion of the developed approach and concluding remarks are presented in Sections V and VI.

II. Mathematical conventions

Given the complexities which can arise from dynamical systems, rigorous mathematical notation must be presented.

1) Variable Naming Conventions: A consistent vector notation is used to represent vector quantities. Boldfaced variables represent vector quantities, unless enclosed by square brackets [ ] which represent matrices.

Vectors with physical interpretations, such as kinematic and kinetic vectors, are accented by a right pointing arrow '→'. Kinematic vector quantities are used to represent displacements, velocities, and accelerations of bodies and points in space, while kinetic vectors represent forces and moments applied to bodies.

The kinematic vector

$$ {}^{c}_{d}\vec{\boldsymbol{a}}^{b} \tag{1}$$

is interpreted as the kinematic vector quantity $\vec{\boldsymbol{a}}$ of body b with respect to c, expressed in frame of reference d. The vector quantity is typically $\boldsymbol{r}$ for translational displacement, $\boldsymbol{v}$ for translational velocity, $\boldsymbol{a}$ for translational acceleration, $\boldsymbol{\omega}$ for angular velocity, $\boldsymbol{\alpha}$ for angular acceleration, and $\boldsymbol{g}$ for gravitational acceleration.

The kinetic vector

$$ {}^{g}_{h}\vec{\boldsymbol{e}}^{f} \tag{2}$$

is interpreted as the kinetic vector $\vec{\boldsymbol{e}}$ of f acting on g, expressed in frame of reference h. The vector quantity is typically $\boldsymbol{F}$ for force vectors and $\boldsymbol{M}$ for moment vectors.

All physical vector quantities are in the form of 3x1 column vectors, where the first, second, and third entries are the vector components along the x, y, and z 3D Cartesian axes respectively. The axes obey the Right-Hand-Rule (RHR) for $x-y-z$ axis orthogonality, and the RHR for angular quantity direction about the z axis. For example, a z axis projecting outwards from the page, has a counter-clockwise positive angular direction.

2) Vector Quantity Time Derivatives: The time differentiated vector quantity is denoted

$$\frac{\mathrm{d}}{\mathrm{d}t}({}^{c}_{d}\vec{\boldsymbol{a}}^{b}) = {}^{c}_{d}\,\dot{\vec{\boldsymbol{a}}}^{b} \tag{3}$$

For this work, the differentiation is performed in the frame of reference of the differentiated vector, and is referred to as the locally-evaluated time derivative. This is significant, as obtaining real accelerations for the dynamic system requires the local time differentiations be evaluated in an inertial frame of reference. Thus the relative position vector quantities, defining the position of a body through multiple frames of reference, must all be expressed with respect to the inertial frame before differentiation can take place.

3) Cross Product Identity: Cross products appear in many instances of kinematic and dynamic derivations, for example: in locally evaluated time derivatives, and moment evaluations due to translational forces. Per Nikravesh [8], cross product evaluation for 3x1 vectors can be expressed as a matrix multiplication using the skew-symmetric matrix operator $\widetilde{[\ ]}$:

$$\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}} = \widetilde{\vec{\boldsymbol{a}}}\vec{\boldsymbol{b}} = -\widetilde{\vec{\boldsymbol{b}}}\vec{\boldsymbol{a}} \tag{4}$$

4) Rotational Transformation Matrix: The kinematic chains of frames of reference used in this paper require the transformation of vector quantities between frames of reference. These transformation matrices are constructed using Euler parameters which define the rotation of angle ϕ about a unit vector $\hat{\boldsymbol{u}}$, where

$$e_0 = \cos\left(\frac{\phi}{2}\right),\ \vec{\boldsymbol{e}} = \begin{Bmatrix} e_1 \\ e_2 \\ e_3 \end{Bmatrix} = \hat{\boldsymbol{u}}\sin\left(\frac{\phi}{2}\right) \tag{5}$$

The transformation matrix is then

$$[\boldsymbol{T}_{b\leftarrow a}] = 2\begin{bmatrix} {e_0}^2 + {e_1}^2 - \frac{1}{2} & e_1e_2 - e_0e_3 & e_1e_3 + e_0e_2 \\ e_1e_2 + e_0e_3 & {e_0}^2 + {e_2}^2 - \frac{1}{2} & e_2e_3 + e_0e_1 \\ e_1e_3 + e_0e_2 & e_2e_3 + e_0e_1 & {e_0}^2 + {e_3}^2 - \frac{1}{2} \end{bmatrix} \tag{6}$$

where $[\boldsymbol{T}_{b\leftarrow a}]$ is the transformation from frame a to frame b. The Euler parameters used in this transformation specifically represent the orientation of frame b with respect to frame a, expressed in the frame a. For successive frame transformations, pre-multiplication of the consecutive transformation matrices in the chain of frames is required since consecutive frames are described with respect to the axes of the previous frame. For example, the transformation of rotated frame of reference c to non-rotated frame of reference a, passing through rotated frame of reference b is

$$[\boldsymbol{T}_{a\leftarrow c}] = [\boldsymbol{T}_{a\leftarrow b}][\boldsymbol{T}_{b\leftarrow c}] \tag{7}$$

Time differentiation of transformation matrices occurs during the differentiation of position vectors to velocity vectors then to acceleration vectors. The identity given by Nikravesh [8], for the time derivative of a transformation matrix, is:

$$[\dot{\boldsymbol{T}}_{a\leftarrow b}] = \frac{\mathrm{d}}{\mathrm{d}t}([\boldsymbol{T}_{a\leftarrow b}]) = {}^{b}_{b}\tilde{\boldsymbol{\omega}}^{a}[\boldsymbol{T}_{a\leftarrow b}] = [\boldsymbol{T}_{a\leftarrow b}]^{b}_{a}\tilde{\boldsymbol{\omega}}^{a} \tag{8}$$

III. Kane's Method

Kane's method for the generation of a system's EOM begins with Kane's equation; the summation of the active and inertial forces corresponding to a generalized coordinate

$$0 = F_i + F_i^*, \quad i = 1, ..., N_{gc} \tag{9}$$

where Equation 9 is the set of N_{gc} scalar equations summing the total inertial forces F_i^* and active forces F_i, corresponding to the ith generalized coordinate. Equation 9 can be vectorized and written as the system

$$\boldsymbol{0} = \boldsymbol{F} + \boldsymbol{F}^* \tag{10}$$

where, summed over the N_B rigid bodies in the system, the inertial and active force vectors are obtained by

$$\boldsymbol{F} = \sum_{k=1}^{N_B} \left(\left[{}^{N}_{k}\boldsymbol{V}^{k}\right]^T \vec{\boldsymbol{R}}^{k} + \left[{}^{N}_{k}\boldsymbol{W}^{k}\right]^T \vec{\boldsymbol{T}}^{k} \right) \tag{11}$$

$$\boldsymbol{F}^* = \sum_{k=1}^{N_B} \left(\left[{}^{N}_{k}\boldsymbol{V}^{k}\right]^T \vec{\boldsymbol{R}}^{*k} + \left[{}^{N}_{k}\boldsymbol{W}^{k}\right]^T \vec{\boldsymbol{T}}^{*k} \right) \tag{12}$$

where, for the kth body, $\vec{\boldsymbol{R}}$ and $\vec{\boldsymbol{T}}$ are the resulting active force and active torque, $\vec{\boldsymbol{R}}^*$ and $\vec{\boldsymbol{T}}^*$ are the translational and angular inertias, and $[\boldsymbol{V}]$ and $[\boldsymbol{W}]$ are the translational and angular partial velocity matrices, respectively. The partial velocity matrices are obtained from the compact expression of body velocities and accelerations in terms of the vector of generalized speeds $\boldsymbol{u}$ and vector of generalized speed time derivatives $\dot{\boldsymbol{u}}$ where, for the kth body in the system, we have

$$
\begin{aligned}
{}^{N}_{k}\vec{\boldsymbol{v}}^{k} &= \left[{}^{N}_{k}\boldsymbol{V}^{k}\right]\boldsymbol{u} && (13)\\
{}^{N}_{k}\vec{\boldsymbol{\omega}}^{k} &= \left[{}^{N}_{k}\boldsymbol{W}^{k}\right]\boldsymbol{u} && (14)\\
{}^{N}_{k}\vec{\boldsymbol{a}}^{k} &= \left[{}^{N}_{k}\boldsymbol{V}^{k}\right]\dot{\boldsymbol{u}} + \vec{\boldsymbol{Z}}_{k} && (15)\\
{}^{N}_{k}\vec{\boldsymbol{\alpha}}^{k} &= \left[{}^{N}_{k}\boldsymbol{W}^{k}\right]\dot{\boldsymbol{u}} + \vec{\boldsymbol{Y}}_{k} && (16)
\end{aligned}
$$

The partial velocity matrices represent the contribution of the system's vector of generalized speeds $\boldsymbol{u}$, and vector of generalized speed time derivatives $\dot{\boldsymbol{u}}$ to the kth body velocity and acceleration, respectively. Coined by Stoneking [9], the terms $\vec{\boldsymbol{Z}}_k$ and $\vec{\boldsymbol{Y}}_k$ are the 'remainder translational accelerations' and 'remainder angular accelerations', respectively, and include all kinematic terms which do not include the generalized speed time derivatives. Notably, Equations 13 and 15, and 14 and 16, have identical partial velocity matrices for velocity and acceleration expressions of a body.

Likewise, the transposes of the partial velocity matrices are used to project the forces acting on the N_B bodies into the permissible motions subspace defined by the set of system generalized coordinates $\boldsymbol{q}$ by pre-multiplication of the force vectors by the transpose of the partial velocity matrix as shown in Equations 11 and 12 [9].

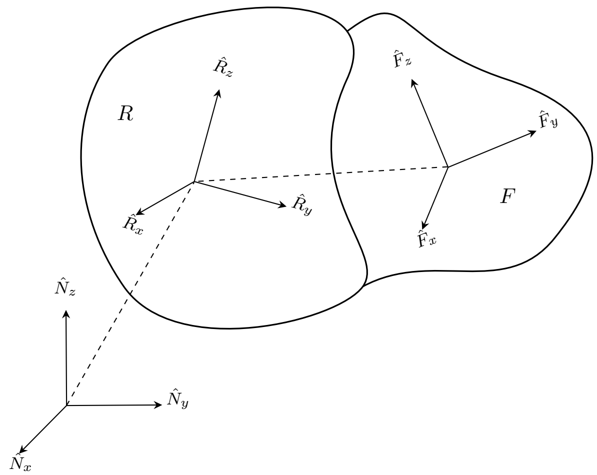


Fig. 1: Rigid body R and flexible body F.

For a single body k, Equation 10 can be expanded

$$
\begin{aligned}
\boldsymbol{0} = {} & \left[{}^{N}_{k}\boldsymbol{W}^{k}\right]^T \Big\{ \vec{\boldsymbol{T}}^{k} - \left[{}_{k}\boldsymbol{I}^{k}\right] \left(\left[{}^{N}_{k}\boldsymbol{W}^{k}\right]\dot{\boldsymbol{u}} + \vec{\boldsymbol{Y}}_{k} \right) \\
& - \left({}^{N}_{k}\tilde{\boldsymbol{\omega}}^{k} \left[{}_{k}\boldsymbol{I}^{k}\right] {}^{N}_{k}\vec{\boldsymbol{\omega}}^{k} \right) \Big\} \\
& + \left[{}^{N}_{k}\boldsymbol{V}^{k}\right]^T \Big\{ (\vec{\boldsymbol{R}}^{k} - m_k \left(\left[{}^{N}_{k}\boldsymbol{V}^{k}\right]\dot{\boldsymbol{u}} + \vec{\boldsymbol{Z}}_{k} \right) \Big\}
\end{aligned} \tag{17}
$$

where m_k is the mass of the kth body, and $\left[{}_{k}\boldsymbol{I}^{k}\right]$ is the rotational inertia matrix for the kth body about its centre of mass (COM), expressed in the kth body's frame. The development of Kane's method in Equations 11-17 is expressed in the respective body frames which ensures that the angular inertia tensor is time-invariant.

The vector of unknown generalized speed time derivatives $\dot{\boldsymbol{u}}$ in Equation 17 is common to all bodies in the system, and when Kane's equation is summed over the N_B bodies in the system, it facilitates the manipulation of the system into the linear first-order form

$$[\boldsymbol{M}_{sys}]\,\dot{\boldsymbol{u}} = \boldsymbol{F}_{sys} \tag{18}$$

which lends itself to incremental state-propagation algorithms.

IV. Rigid-Flexible System Formulation For Kane's Method

The development of the formulation begins with the separation of the rigid-flexible system into coupled rigid and flexible bodies R and F. They are configured into an open chain form with respect to the inertial frame N as shown in Figure 1. For the coupled rigid and flexible bodies, Equations 11 and 12 become

$$
\begin{aligned}
\boldsymbol{F} = {} & \left[{}^{N}_{R}\boldsymbol{V}^{R}\right]^T \vec{\boldsymbol{R}}^{R} + \left[{}^{N}_{R}\boldsymbol{W}^{R}\right]^T \vec{\boldsymbol{T}}^{R} \\
& + \left[{}^{N}_{F}\boldsymbol{V}^{F}\right]^T \vec{\boldsymbol{R}}^{F} + \left[{}^{N}_{F}\boldsymbol{W}^{F}\right]^T \vec{\boldsymbol{T}}^{F}
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
\boldsymbol{F}^* = {} & \left[{}^{N}_{R}\boldsymbol{V}^{R}\right]^T \vec{\boldsymbol{R}}^{*R} + \left[{}^{N}_{R}\boldsymbol{W}^{R}\right]^T \vec{\boldsymbol{T}}^{*R} \\
& + \left[{}^{N}_{F}\boldsymbol{V}^{F}\right]^T \vec{\boldsymbol{R}}^{*F} + \left[{}^{N}_{F}\boldsymbol{W}^{F}\right]^T \vec{\boldsymbol{T}}^{*F}
\end{aligned} \tag{20}
$$

The rigid body is immediately treatable; however, internal inertial, elastic, and viscous effects must be considered in the transient deformation of the flexible body undergoing large reference motions.

For a variety of reasons, it may be desirable to represent the flexible body using standard finite element methods. For small nodal displacements and rotations, the linear dynamic finite element (DFE) model is suitable, and generally takes the form

$$[\boldsymbol{M}]\,\ddot{\boldsymbol{x}}_n + [\boldsymbol{C}]\,\dot{\boldsymbol{x}}_n + [\boldsymbol{K}]\,\boldsymbol{x}_n = \boldsymbol{F}(\dot{\boldsymbol{x}}, \boldsymbol{x}_n) \tag{21}$$

where $[\boldsymbol{M}]$ is the mass matrix, $[\boldsymbol{C}]$ is the damping matrix, $[\boldsymbol{K}]$ is the stiffness matrix, $\boldsymbol{F}(\dot{\boldsymbol{x}}_n, \boldsymbol{x}_n)$ is the nodal forcing vector, and $\ddot{\boldsymbol{x}}_n$, $\dot{\boldsymbol{x}}_n$, $\boldsymbol{x}_n$ are the nodal accelerations, velocities and deflections, respectively [10].

Yet, the discretization of the flexible body to obtain approximate behaviour in terms of nodal responses is not readily incorporated into Kane's method as presented above since it is expressed for a system of rigid bodies. To incorporate finite element modelling into Kane's method, the application of Equations 11 and 12 specifically to rigid bodies must be relaxed. Kane's Equation 9 is applied to the finite element nodes such that the nodal displacements, velocities, and accelerations are defined using generalized coordinates, speeds, and speed time derivatives, respectively. Then, the response of the flexible body is approximated by finite element modelling applied using Kane's method.

Naturally, it would be convenient to independently develop the global DFE mass, damping, and stiffness matrices using standard techniques for Kane's method, rather than exhaustively considering every coupling force on every finite element node. As will be made evident in the following sections, with an appropriate interpretation of the frame of reference definitions similar to the IECS approach, independent DFE matrix development can be achieved. The specific definitions required for the presented formulation fall under three categories:

1) Frame of reference definitions;
2) Expression of nodal accelerations, velocities, and displacements; and
3) Separation of the nodal forces vector into known externally-applied and unknown coupling forces.

A. Frames of reference

The frame of reference definitions follow from Shabana's IECS approach to constrained motion of deformable bodies [7], [11]. The frame definitions of the IECS method are interpreted such that they satisfy both Kane's method and DFE modelling requirements. These frames of reference are presented in Figure 2.

The four required frames are set up in the open chain $N - R - G - n_j$ where a frame position and orientation is defined relative to the previous frame in the chain, with independent branches for each n_j frame stemming from a common G frame.

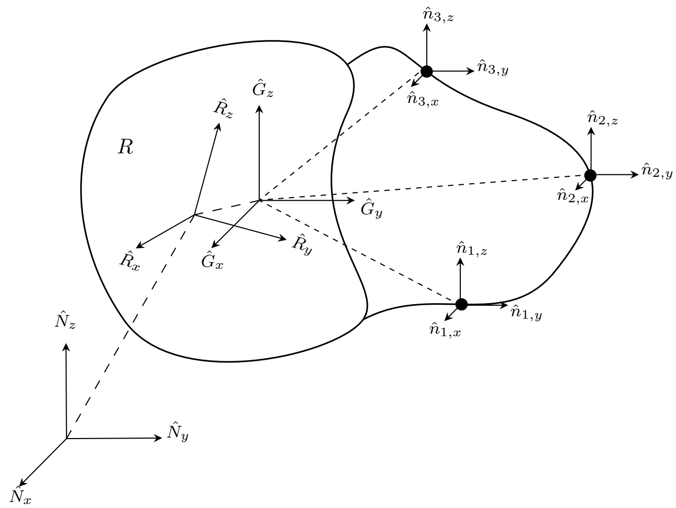


Fig. 2: Rigid-flexible body frames of reference. The generic body in this figure is considered to have a three-node finite element representation.

The first frame is the inertial frame N which serves as the unique non-accelerating reference relative to which the whole dynamical system is defined.

The second frame is the rigid body frame of reference R with an origin coincident with the COM of the rigid body.

The third frame G serves as the intermediate frame between the n_j and R frames. Its origin is rigidly fixed to the origin of the R frame, and is axis-aligned to the undeformed n_j frame. The G frame is equivalent to the *intermediate element coordinate system* of the IECS method in [7]. A notable deviation from the IECS definition is that all elements share the intermediate G frame. Nevertheless all elements retain independent transformations from the local coordinates to the G frame. This satisfies the frame alignment requirement for arbitrarily-oriented element assembly in the FEM. The G frame adopts the name 'Global' frame from finite element assembly nomenclature.

The fourth frame of reference n_j is the jth DFE node frame of reference and is fixed to the nodes of a finite element. All nodal frames are axis-aligned in the undeformed state.

For this open chain of frames of reference as shown in Figure 3, the generalized coordinates q and speeds u are carefully chosen. The natural Cartesian coordinates and speeds which characterize a frame relative to the previous frame are prescribed to be the translational generalized coordinates generalized speeds. The angular generalized coordinates of one frame with respect to the previous frame are represented using four Euler parameters to avoid singularities in rotation matrices, and the three angular velocity components are chosen as the generalized angular speeds. In the context of state propagation, the provided definition of angular quantities is convenient. Nikravesh [8] provides a direct relationship between the angular velocities and Euler parameters.

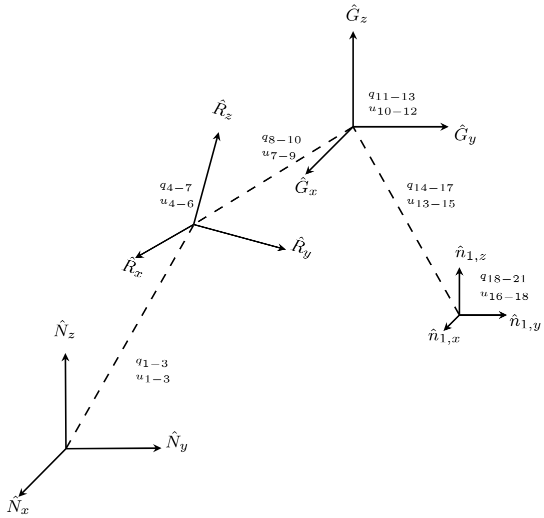


Fig. 3: Generalized coordinates $\boldsymbol{q}$ and speeds $\boldsymbol{u}$. Each frame is described relative to the previous frame in the chain by a pair of translational and angular vectors containing generalized coordinates and speeds. In this figure, only the open chain branch to frame n_1 is shown for clarity.

B. Kinematic Quantities

To enable independent modelling of the finite element matrices, the nodal accelerations, speeds, and deflections must be expressed in terms of the generalized speed time derivatives $\dot{\boldsymbol{u}}$, speeds $\boldsymbol{u}$, and coordinates $\boldsymbol{q}$ as required for Kane's method.

The nodal deflections $\boldsymbol{x}_n$ from Equation 21, expressed in the global finite element frame G are

$$\boldsymbol{x}_n = \begin{Bmatrix} \boldsymbol{x}_{n_1} \\ \vdots \\ \boldsymbol{x}_{n_j} \end{Bmatrix} = \begin{Bmatrix} {}^G_G\vec{\boldsymbol{r}}^{\,n_1} - {}^G_G\vec{\boldsymbol{r}}_0^{\,n_1} \\ {}^G_G\boldsymbol{\theta}^{\,n_1} \\ \vdots \\ {}^G_G\vec{\boldsymbol{r}}^{\,n_j} - {}^G_G\vec{\boldsymbol{r}}_0^{\,n_j} \\ {}^G_G\boldsymbol{\theta}^{\,n_j} \end{Bmatrix} = \boldsymbol{q}_n - \boldsymbol{q}_{n,0} \tag{22}$$

where ${}^G_G\vec{\boldsymbol{r}}_0^{\,n_j}$ is the undeformed position of the jth DFE node relative to the global DFE frame G, expressed in G, composed of the undeformed translational generalized coordinates. The difference between the undeformed and deformed translational generalized coordinates represents the nodal deflections in the G frame. The Euler parameters representing the orientation of frame n_j are converted to quantities representative of the nodal slopes ${}^G_G\boldsymbol{\theta}^{\,n_j}$. In FEM, the slopes are expressed relative to, and in the global assembly frame G. Therefore, the Euler parameters can be converted to a extrinsic rotation convention to obtain the nodal slopes. The relationship between the Euler parameters in Equations 5 and nodal slopes represented as extrinsic Tait-Bryan angles is

$$\begin{aligned} {}^G_G\boldsymbol{\theta}^{\,n_j} &= \begin{Bmatrix} \theta_x \\ \theta_y \\ \theta_z \end{Bmatrix} \\ &= \begin{Bmatrix} \text{atan2}\left(2(e_2e_3 + e_0e_1), \frac{1}{2} - ({e_1}^2 + {e_2}^2)\right) \\ \arcsin\left(2(e_1e_3 - e_0e_2)\right) \\ \text{atan2}\left(2(e_1e_2 + e_0e_3), \frac{1}{2} - ({e_2}^2 + {e_3}^2)\right) \end{Bmatrix} \end{aligned} \tag{23}$$

where θ_x, θ_y, θ_z are the slopes about the x, y, and z axes of the G frame respectively [12]. In the undeformed configuration these slopes are zero. Additionally, Tait-Bryan angles are chosen since rotation is described using the three orthogonal axes, in contrast to Euler angles which use only two of the available axes in rotation.[1]

The vector of nodal velocities $\dot{\boldsymbol{x}}_n$ can be expressed in terms of the compact velocity expressions found in Equations 13 and 14 such that

$$\dot{\boldsymbol{x}}_n = \begin{bmatrix} \left[{}^N_G\boldsymbol{V}^{\,n_1}\right] \\ \left[{}^N_G\boldsymbol{W}^{\,n_1}\right] \\ \vdots \\ \left[{}^N_G\boldsymbol{V}^{\,n_j}\right] \\ \left[{}^N_G\boldsymbol{W}^{\,n_j}\right] \end{bmatrix} \boldsymbol{u} = \left[{}^N_G\boldsymbol{V}^{\,n}\right]\boldsymbol{u} \tag{24}$$

where $\left[{}^N_G\boldsymbol{V}^{\,n}\right]$ is the nodal partial velocities matrix expressed in the G frame. Similarly to the nodal velocities, the vector of nodal linear and angular accelerations can be expressed in the form of Equations 15 and 16 such that

$$\ddot{\boldsymbol{x}}_n = \left[{}^N_G\boldsymbol{V}^{\,n}\right]\dot{\boldsymbol{u}} + \boldsymbol{Z}_n \tag{25}$$

where $\boldsymbol{Z}_n$ is the vector of nodal remainder accelerations terms for all DFE nodes, and takes the form

$$\boldsymbol{Z}_n = \begin{Bmatrix} \vec{\boldsymbol{Z}}_{n_1} \\ \vec{\boldsymbol{Y}}_{n_1} \\ \vdots \\ \vec{\boldsymbol{Z}}_{n_j} \\ \vec{\boldsymbol{Y}}_{n_j} \end{Bmatrix} \tag{26}$$

Substituting Equations 22, 24, and 25 into Equation 21 yields

$$[\boldsymbol{M}]\left(\left[{}^N_G\boldsymbol{V}^{\,n}\right]\dot{\boldsymbol{u}} + \boldsymbol{Z}_n\right) + [\boldsymbol{C}]\left[{}^N_G\boldsymbol{V}^{\,n}\right]\boldsymbol{u} + [\boldsymbol{K}]\left(\boldsymbol{q}_n - \boldsymbol{q}_{n,0}\right) = {}^n_G\boldsymbol{F} \tag{27}$$

Here, the DFE model is assembled in the intermediate global finite element frame G with axis-aligned node frames n_j (in the undeformed configuration) as required by the procedure for the assembly of arbitrarily-oriented finite elements. Discussion in Section V will show that exact moments of mass and mass moments of inertia are preserved.

[1]At a glance, the relationships expressed by Equation 23 appear to be inefficient in the context of numerical simulations due to inverse trigonometric functions. In the case of a rigid-flexible chain, it may be more efficient to directly prescribe nodal deflections and slopes as generalized coordinates since transformations to body frames are not required for the finite element representation of flexible bodies due to the time-variant inertia tensor. However, in a more general case such as a rigid-flexible-rigid chain with hinged interface configurations, using Euler parameters would circumvent 'gimbal-locked' configurations under large relative rotational motion.

Now Equation 27 can be separated into the active and inertial forces and torques for Equations 19 and 20:

$$\begin{aligned}\boldsymbol{R}^F + \boldsymbol{T}^F = {}^n_G\boldsymbol{F} &- [\boldsymbol{C}]\left[{}^N_G\boldsymbol{V}^n\right]\boldsymbol{u} \\ &- [\boldsymbol{K}](\boldsymbol{q}_n - \boldsymbol{q}_{n,0})\end{aligned} \tag{28}$$

$$\boldsymbol{R}^{*F} + \boldsymbol{T}^{*F} = -[\boldsymbol{M}]\left(\left[{}^N_G\boldsymbol{V}^n\right]\dot{\boldsymbol{u}} + \boldsymbol{Z}_n\right) \tag{29}$$

Since the translational and angular inertial, damping, and elastic forces are grouped by the use of mass, damping, and stiffness matrices, the forces and torques are grouped in Equations 28 and 29. It follows that the premultiplication by the transpose of the translational and angular partial velocity matrices in Equations 19 and 20 is performed by the transpose $\left[{}^N_G\boldsymbol{V}^n\right]^T$ as will be shown in Subsection IV-D.

C. Known and Unknown Node Forces

The separation of the rigid-flexible system into coupled rigid and flexible bodies using the FSM approach implies internal coupling forces. Additionally, finite element modelling of the flexible body requires application of forces at the DFE nodes. Therefore the coupling forces at the interface between the rigid and flexible bodies occurs at the DFE nodes. The nodal forces vector must then be separated into the known externally-applied nodal forces and the unknown internal coupling forces such that

$${}^n_G\boldsymbol{F} = {}^n_G\boldsymbol{F}_k + {}^n_G\boldsymbol{F}_u \tag{30}$$

where $\boldsymbol{F}_k$ is the vector of known externally-applied nodal forces and moments, and $\boldsymbol{F}_u$ is the vector of unknown coupling nodal forces and moments. The coupling forces must also be expressed as equal but opposed forces and moments acting on the rigid body as the coupling forces may not be directly acting at the COM of the rigid body. The unknown coupling force ${}^R_R\vec{\boldsymbol{F}}_u$ acting on the rigid body is expressed

$${}^R_R\vec{\boldsymbol{F}}_u = -\sum_j [\boldsymbol{T}_{R\leftarrow G}]\,{}^{n_j}_G\vec{\boldsymbol{F}}_u \tag{31}$$

where the total coupling force acting on the rigid body is the summation over all DFE nodes. For nodes not mechanically joined to the rigid body, their coupling force magnitude is zero.

The unknown coupling moment ${}^R_R\vec{\boldsymbol{M}}_u$ acting on the rigid body is expressed

$${}^R_R\vec{\boldsymbol{M}}_u = -\sum\left({}^R_R\widetilde{\boldsymbol{r}}^{\,n_j}[\boldsymbol{T}_{R\leftarrow G}]\,{}^{n_j}_G\vec{\boldsymbol{F}}_u + [\boldsymbol{T}_{R\leftarrow G}]\,{}^{n_j}_G\vec{\boldsymbol{M}}_u\right) \tag{32}$$

where the total coupling moment acting on the rigid body is the summation over all DFE. Likewise for nodes not mechanically joined to the rigid body, their coupling moment magnitude is zero.

To maintain full rank of the final set of EOM for the rigid-flexible body, the unknown coupling forces and moments acting on the rigid body must be expressed in terms of the unknown nodal coupling forces. Let $[\boldsymbol{\tau}_{R\leftarrow G}]$ be the coupling force matrix which converts the unknown coupling nodal forces ${}^n_G\boldsymbol{F}_u$ into equal but opposed forces ${}^R_R\vec{\boldsymbol{F}}_u$ and moments ${}^R_R\vec{\boldsymbol{M}}_u$ acting on the rigid body, expressed in the rigid body's frame of reference R such that

$$\begin{Bmatrix}{}^R_R\vec{\boldsymbol{F}}_u \\ {}^R_R\vec{\boldsymbol{M}}_u\end{Bmatrix} = -[\boldsymbol{\tau}_{R\leftarrow G}]\,{}^n_G\boldsymbol{F}_u \tag{33}$$

By inspection of Equations 31 and 32, $[\boldsymbol{\tau}_{R\leftarrow G}]$ is defined

$$\begin{aligned}&[\boldsymbol{\tau}_{R\leftarrow G}] = \\ &\begin{bmatrix}[\boldsymbol{T}_{R\leftarrow G}] & [0]_{3\times 3} & \dots & [\boldsymbol{T}_{R\leftarrow G}] & [0]_{3\times 3} \\ {}^R_R\widetilde{\boldsymbol{r}}^{\,n_j}[\boldsymbol{T}_{R\leftarrow G}] & [\boldsymbol{T}_{R\leftarrow G}] & \dots & {}^R_R\widetilde{\boldsymbol{r}}^{\,n_j}[\boldsymbol{T}_{R\leftarrow G}] & [\boldsymbol{T}_{R\leftarrow G}]\end{bmatrix}\end{aligned} \tag{34}$$

Columns of $[\boldsymbol{\tau}_{R\leftarrow G}]$ which correspond to unconstrained nodal degrees of freedom are zero. This includes nodes which are mechanically joined to the rigid body. This property can be used to characterize different rigid-flexible interfaces. For example, the columns corresponding to coupling moment degrees of freedom are equivalently zero for hinged interfaces. For systems with multiple rigid bodies coupled to flexible bodies, there are independent coupling force matrices.

D. Dynamic Finite Element Incorporation

With the provided definitions, Kane's method for a rigid-flexible system with DFE modelling of the flexible body and direct incorporation of mass, damping, and stiffness matrices yields the formulation

$$\begin{aligned}\boldsymbol{F}^R = &\left[{}^N_R\boldsymbol{V}^R\right]^T\vec{\boldsymbol{R}}^R + \left[{}^N_R\boldsymbol{W}^R\right]^T\vec{\boldsymbol{T}}^R \\ &- \left[\left[{}^N_R\boldsymbol{V}^R\right]^T\left[{}^N_R\boldsymbol{W}^R\right]^T\right][\boldsymbol{\tau}_{R\leftarrow G}]\,{}^n_G\boldsymbol{F}_u\end{aligned} \tag{35}$$

$$\begin{aligned}\boldsymbol{F}^F = \left[{}^N_G\boldsymbol{V}^n\right]^T&\Big({}^n_G\boldsymbol{F}_k + {}^n_G\boldsymbol{F}_u - [\boldsymbol{C}]\left[{}^N_G\boldsymbol{V}^n\right]\boldsymbol{u} \\ &- [\boldsymbol{K}](\boldsymbol{q}_n - \boldsymbol{q}_{n,0})\Big)\end{aligned} \tag{36}$$

$$\begin{aligned}\boldsymbol{F}^{*R} = &-\left[{}^N_R\boldsymbol{V}^R\right]^T m_R\left(\left[{}^N_R\boldsymbol{V}^R\right]\dot{\boldsymbol{u}} + \vec{\boldsymbol{Z}}_R\right) \\ &- \left[{}^N_R\boldsymbol{W}^R\right]^T\Big\{\left[{}_R\boldsymbol{I}^R\right]\left(\left[{}^N_R\boldsymbol{W}^R\right]\dot{\boldsymbol{u}} + \vec{\boldsymbol{Y}}_R\right) \\ &+ \left({}^N_R\widetilde{\boldsymbol{\omega}}^R\left[{}_R\boldsymbol{I}^R\right]{}^N_R\vec{\boldsymbol{\omega}}^R\right)\Big\}\end{aligned} \tag{37}$$

$$\boldsymbol{F}^{*F} = -\left[{}^N_G\boldsymbol{V}^n\right]^T\left([\boldsymbol{M}]\left(\left[{}^N_G\boldsymbol{V}^n\right]\dot{\boldsymbol{u}} + \boldsymbol{Z}_n\right)\right) \tag{38}$$

The set of Equations 35-38 is assembled into the form of Equation 10 to obtain the final formulation of Kane's method for the EOM of a rigid-flexible body, where finite element matrices can be incorporated directly, as

$$\begin{aligned}\boldsymbol{0} = &\left[{}^N_R\boldsymbol{V}^R\right]^T\vec{\boldsymbol{R}}^R + \left[{}^N_R\boldsymbol{W}^R\right]^T\vec{\boldsymbol{T}}^R \\ &+ \left[{}^N_G\boldsymbol{V}^n\right]^T\left({}^n_G\boldsymbol{F}_k - [\boldsymbol{C}]\left[{}^N_G\boldsymbol{V}^n\right]\boldsymbol{u} - [\boldsymbol{K}](\boldsymbol{q}_n - \boldsymbol{q}_{n,0})\right) \\ &+ \left(\left[{}^N_G\boldsymbol{V}^n\right]^T - \left[\left[{}^N_R\boldsymbol{V}^R\right]^T\left[{}^N_R\boldsymbol{W}^R\right]^T\right][\boldsymbol{\tau}_{R\leftarrow G}]\right){}^n_G\boldsymbol{F}_u \\ &- \left[{}^N_R\boldsymbol{V}^R\right]^T m_R\left(\left[{}^N_R\boldsymbol{V}^R\right]\dot{\boldsymbol{u}} + \vec{\boldsymbol{Z}}_R\right) \\ &- \left[{}^N_R\boldsymbol{W}^R\right]^T\Big\{\left[{}_R\boldsymbol{I}^R\right]\left(\left[{}^N_R\boldsymbol{W}^R\right]\dot{\boldsymbol{u}} + \vec{\boldsymbol{Y}}_R\right) \\ &+ \left({}^N_R\widetilde{\boldsymbol{\omega}}^R\left[{}_R\boldsymbol{I}^R\right]{}^N_R\vec{\boldsymbol{\omega}}^R\right)\Big\} \\ &- \left[{}^N_G\boldsymbol{V}^n\right]^T\left([\boldsymbol{M}]\left(\left[{}^N_G\boldsymbol{V}^n\right]\dot{\boldsymbol{u}} + \boldsymbol{Z}_n\right)\right)\end{aligned} \tag{39}$$

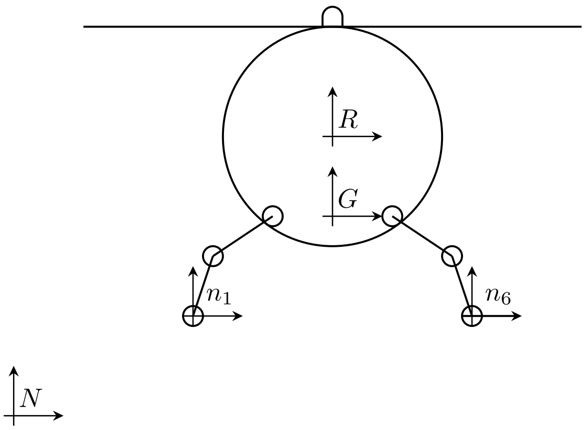
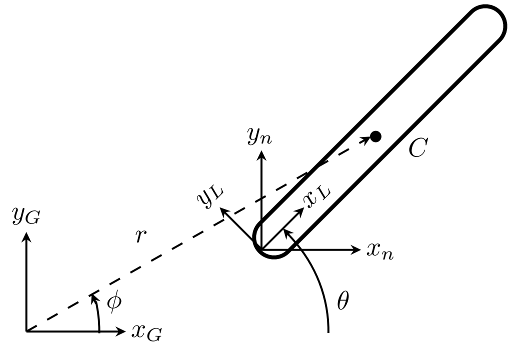


Fig. 4: SRAMSS-2D rigid-flexible aircraft system frame configuration.



0

Fig. 5: Planar beam coordinate systems.

Equation 39 can be manipulated into a linear system similar to the form of Equation 18 where a vector of unknown generalized speed time derivatives and unknown internal coupling forces $\{\dot{\boldsymbol{u}} \quad {}^{n}_{G}\boldsymbol{F}_u\}^T$ can be solved. The rank of the system does not change due to the addition of unknown coupling forces since they correspond generalized coordinates of the interface degrees of freedom known to have zero relative acceleration. This property highlights the utility of using the chain of frames in combination with generalized coordinates describing relative configuration to the previous frame in the system.

V. Discussion

The formulation presented in Equation 39 presents a practical approach to the generation of the EOM for a rigid-flexible system. Moreover, the formulation can be extended to multiple independent rigid-flexible systems in the same overall system where individual branches in the open chain of frames are added for each rigid-flexible system. The practicality of the presented formulation is underscored by the independent assembly of the linear finite element matrices using standard methods, and the systematic generation of the required coefficients by inspection of the kinematics for the system.

The planar equivalent of the presented formulation has been successfully used in the development of SRAMSS-2D (Skid-equipped Rotary-wing Aircraft Manoeuvring and Securing Simulation) where the response of ship-embarked rotary-wing aircraft is simulated [13]. The aircraft is modelled using a rigid body to represent the airframe, while a dynamic finite beam element model is used to model the compliant skid-tube landing gear as shown in Figure 4.

In its current configuration, the presented formulation was developed with the intended purpose of incorporating linear DFE modelling which uses cubic interpolation shape functions to develop the consistent mass matrix and stiffness matrix for the finite element beam. For large nodal deformations and finite nodal rotations, these standard techniques fail to accurately capture non-linear phenomena. Nevertheless, like the IECS method, exact rigid-body mass moment of inertia and first moment of mass are preserved. Figure 5 shows a planar uniform six degree-of-freedom slender beam element with COM C positioned at r and orientation θ with respect to the global frame G. The beam has length l, cross-sectional area A, mass density ρ, mass $m = \rho A l$, and the deflection at an arbitrary point along the beam is characerized by shape function $\boldsymbol{S}$ as

$$\boldsymbol{S} = \begin{bmatrix} \frac{-1}{l} & 0 & 0 & \frac{1}{l} & 0 & 0 \\ 0 & -\frac{6}{l^2}+\frac{12x}{l^3} & -\frac{4}{l}+\frac{6}{l^2} & 0 & \frac{6}{l^2}-\frac{12x}{l^3} & -\frac{2}{l}+\frac{6}{l^2} \end{bmatrix} \tag{40}$$

For the beam in Figure 5, the consistent mass matrix, expressed in local beam coordinates L, is

$$[\boldsymbol{M}_L] = \frac{\rho A l}{420}\begin{bmatrix} 140 & 0 & 0 & 70 & 0 & 0 \\ 0 & 156 & 22l & 0 & 54 & -13l \\ 0 & 22l & 4l^2 & 0 & 13l & -3l^2 \\ 70 & 0 & 0 & 140 & 0 & 0 \\ 0 & 54 & 13l & 0 & 156 & -22l \\ 0 & -13l & -3l^2 & 0 & -22l & 4l^2 \end{bmatrix} \tag{41}$$

When transformed to the n frames which are axis-aligned to the G frame in the undeformed configuration for assembly using standard techniques, the mass matrix becomes

$$[\boldsymbol{M}_G] = [\boldsymbol{M}_n] = [\boldsymbol{T}_{L\leftarrow n}]^T [\boldsymbol{M}_L] [\boldsymbol{T}_{L\leftarrow n}] \tag{42}$$

Described in local coordinates, the COM C of the beam at $x_L{=}l/2$ is given

$$\boldsymbol{x}_{L,C} = \begin{bmatrix} -\frac{l}{2} & 0 & 0 & \frac{l}{2} & 0 & 0 \end{bmatrix}^T \tag{43}$$

When transformed to the n frame, the beam is expressed

$$\boldsymbol{x}_{n,C} = \begin{bmatrix} -\frac{l}{2}\cos\theta & -\frac{l}{2}\sin\theta & 0 & \frac{l}{2}\cos\theta & \frac{l}{2}\sin\theta & 0 \end{bmatrix}^T \tag{44}$$

The mass moment of inertia J_C evaluated at the COM in the n frame is

$$J_C = \boldsymbol{x}_{n,C}^T [\boldsymbol{M}_n] \boldsymbol{x}_{n,C} = \frac{ml^2}{12} \tag{45}$$

which is the mass moment of inertia definition for a beam about its COM. The first moment of mass M_C is also equivalently zero at the COM

$$M_C = [\boldsymbol{M}_n]\, \boldsymbol{x}_{n,C} \equiv 0 \tag{46}$$

More generally, for a single beam in a discrete representation of a flexible body using the G frame as the common reference, the mass moment of inertia property still holds. For the beam illustrated in Figure 5, the beam description expressed in the G frame is

$$\boldsymbol{x}_{G,C} = \begin{bmatrix} r\cos\phi - \frac{l}{2}\cos\theta \\ r\sin\phi - \frac{l}{2}\sin\theta \\ 0 \\ r\cos\phi + \frac{l}{2}\cos\theta \\ r\sin\phi + \frac{l}{2}\sin\theta \\ 0 \end{bmatrix} \tag{47}$$

The mass moment of inertia J_C evaluated at the COM in the G frame is then

$$J_C = \boldsymbol{x}_{G,C}^T\, [\boldsymbol{M}_n]\, \boldsymbol{x}_{G,C} = \frac{ml^2}{12} + mr^2 \tag{48}$$

which is the mass moment of inertia definition for a beam rotating about a parallel axis.
These results obtained for the planar example above can easily be generalized to obtain similar results for the 3D beam element with 12 degrees of freedom.

The choice of linear FEM imply certain limitations to rigid-flexible modelling capabilities of the presented formulation in its current form. While the linear dynamic finite element model is perfectly adequate for elements with infinitesimal rotations, future work will be undertaken to address the modelling of flexible elements with finite nodal deflection and rotation. Inclusion of comprehensive non-=linear deformation and inertial phenomena motivates this goal. Recent research by Nocent and Remion [14], Theetten *et al.* [15], and Valentini and Pennestrì [16] proposse the modelling of flexible beams using dynamic splines. In fact, Sanborn and Shabana [17] propose the possibility of relating a spline formalism with the ANCF approach. Likewise, it suggests the possibility for B-spline formalism integration into the presented formulation to accurately model flexible bodies while retaining independent element matrix development; though it is expected that other non-linear terms may appear as well.

VI. Conclusion

The presented method extends the conventional formulation of multibody dynamics based on Kane's method to enable the practical inclusion of independently-developed finite element representations of flexible bodies. One must only be attentive to the setup of the frames of reference which represent and couple the rigid and flexible bodies in the system. In the final formulation presented, the partial velocities matrices, remainder acceleration terms, and other coefficients can be systematically obtained by inspection of the kinematic development of the system. Then, the EOM can be automatically

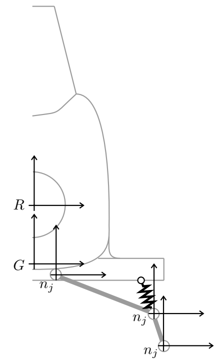


Fig. 6: Planar beam coordinate systems.

generated, and the vector of generalized accelerations can be solved.

Ultimately, the presented method mitigates the laborious developments required by more conventional methods such as Lagrange, Newton-Euler, and the standard application of Kane's method.

The application of this formulation has been verified in the context of shipboard helicopter dynamic interface analysis for aircraft equipped with stiff skid-type landing gear. Future work will aim to expand capabilities in modelling more compliant landing gear. Nevertheless, this formulation is suitable to a wide range of industries where the modelling of flexible structures which undergo large spatial reference motions is required.

References

[1] A. A. Shabana, "Flexible multibody dynamics: Review of past and recent developments," *Multibody System Dynamics*, vol. 1, no. 2, pp. 189–222, Jun 1997.

[2] R. L. Huston, "Multi-body dynamics including the effects of flexibility and compliance," *Computers & Structures*, vol. 14, no. 5, pp. 443–451, 1981.

[3] ——, "Computer methods in flexible multibody dynamics," *International Journal for Numerical Methods in Engineering*, vol. 32, no. 8, pp. 1657–1668, 1991.

[4] R. L. Huston and Y. Wang, "Flexibility effects in multibody systems," *Computer Aided Analysis of Rigid and Flexible Mechanical Systems*, pp. 351–376, 1994.

[5] T. R. Kane, P. W. Likins, and D. A. Levinson, *Spacecraft Dynamics*. New York: McGraw-Hill, 1983.

[6] A. Purushotham and J. Anjeneyulu, "Kane's Method for Robotic Arm Dynamics: a Novel Approach," *IOSR Journal of Mechanical and Civil Engineering (IOSR-JMCE)*, vol. 6, no. 4, pp. 7–13, May-June 2013.

[7] A. A. Shabana, "Finite Element Incremental Approach and Exact Rigid Body Inertia," *Journal of Mechanical Design*, vol. 118, no. 2, pp. 171–178, 06 1996.

[8] P. E. Nikravesh, *Computer-Aided Analysis of Mechanical Systems*. Englewood Cliffs, NJ: Prentice Hall, 1988.

[9] E. Stoneking, "Implementation of Kane's Method for a Spacecraft Composed of Multiple Rigid Bodies," August 2013.

[10] K.-J. Bathe, *Finite Element Procedures*. New Jersey: Prentice Hall, 1996, p. 768.

[11] A. A. Shabana, "Constrained motion of deformable bodies," *International Journal for Numerical Methods in Engineering*, vol. 32, no. 8, pp. 1813–1831.

[12] P. Berner, "Technical concepts orientation, rotation, velocity and acceleration, and the srm," SEDRIS, June 2008.

[13] A. Schock and R. Langlois, "Spacial simulation of shipboard operations for skid-equipped rotary-wing aircraft," in *Progress in Canadian Mechanical Engineering*, Charlottetown, P.E.I.

[14] O. Nocent and Y. Remion, "Continuous deformation energy for dynamic material splines subject to finite displacements," in *Computer Animation and Simulation 2001*. Vienna: Springer Vienna, 2001, pp. 87–97.

[15] A. Theetten, L. Grisoni, C. Andriot, and B. Barsky, "Geometrically exact dynamic splines," *Computer-Aided Design*, vol. 40, no. 1, pp. 35–48, 2008, constrained Design of Curves and Surfaces.

[16] P. P. Valentini and E. Pennestrì, "Modeling elastic beams using dynamic splines," *Multibody System Dynamics*, vol. 25, no. 3, pp. 271–284, Mar 2011.

[17] G. Sanborn and A. Shabana, "On the integration of computer aided design and analysis using the finite element absolute nodal coordinate formulation," *Multibody System Dynamics*, vol. 22, pp. 181–197, 09 2009.